Lecture 11

Wednesday, September 28, 2016 9:06 AM

3.5 Implicit Differentiation

• Ex Find $y' = \frac{dy}{dx}$

if $\sin(x+y) = y^2 \cos x$.

Idea Diff implicitly wrt x, remembering that y is a function of x.

$$\frac{d}{dx}\left[\sin(x+y)\right] = \frac{d}{dx}\left[y^2 \cos x\right]$$

$$\Rightarrow \cos(x+y) \cdot \frac{d}{dx}[x+y] = y^2 \cdot \frac{d}{dx}[\cos x] + \frac{d}{dx}[y^2] \cos x$$

$$\Rightarrow \cos(x+y) \cdot [1 + y'] = -y^2 \sin x + 2y \cdot y' \cdot \cos x$$

$$\Rightarrow \cos(x+y) + \cos(x+y) \cdot y' = -y^2 \sin x + 2yy' \cos x$$

$$\Rightarrow \cos(x+y) \cdot y' - 2yy' \cos x = -y^2 \sin x - \cos(x+y)$$

$$\Rightarrow y'\left[\cos(x+y) - 2y\cos x\right] = -y^2 \sin x - \cos(x+y)$$

$$\Rightarrow \boxed{y' = \frac{-y^2 \sin x - \cos(x+y)}{\cos(x+y) - 2y \cos x}}$$

Thm f is 1-1 diff. function w/ inverse function f^{-1} and $f'(f^{-1}(a)) \neq 0$.

Then the inverse function is diff @ a and

$$(f^{-1})'(a) = \frac{1}{f'(f^{-1}(a))}$$

ROUGH IDEA

$y = f^{-1}(x)$, want to find $y' = \frac{dy}{dx}$

$$\Downarrow$$

$$f(y) = x \xRightarrow{\text{Imp. Diff}} \frac{d}{dx}[f(y)] = \frac{d}{dx}[x]$$

$$\Rightarrow f'(y) \cdot y' = 1 \Rightarrow y' = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Inverse Trig functions $y = \sin x$, $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$

$y = \sin^{-1}(x)$, $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\Downarrow$$

$\sin y = x$ and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\Downarrow \text{I.D}$$

$$\cos y \cdot y' = 1 \Rightarrow y' = \frac{1}{\cos y}$$

$$\sin^2 u + \cos^2 u = 1 \Rightarrow x^2 + \cos^2 u = 1$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

$\sin^{-1}$

$$= \frac{1}{\cos(\sin^{-1} x)} = \frac{1}{\sqrt{1-x^2}}$$

$\cos y \cdot y' = 1 \Rightarrow y' = \frac{1}{\cos y}$

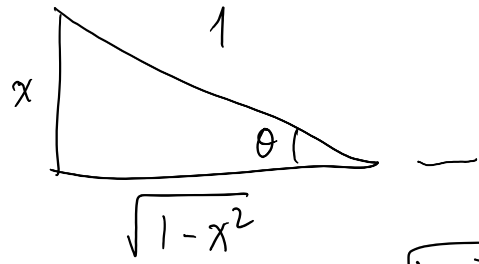
$\sin^2 y + \cos^2 y = 1 \Rightarrow x^2 + \cos^2 y = 1$

$\Rightarrow \cos^2 y = 1 - x^2 \Rightarrow \cos y = \sqrt{1-x^2}$

$\Rightarrow y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$

$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1-x^2}}$

$\cos(\underbrace{\sin^{-1} x}_{\theta})$ $\sqrt{1-x^2}$



$\sin\theta = x$

$\cos\theta = \frac{\sqrt{1-x^2}}{1} = \sqrt{1-x^2}$

DIY

$\frac{d}{dx}[\cos^{-1} x] = -\frac{1}{\sqrt{1-x^2}}$ Rest look up

$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1+x^2}$.

Ex $f(x) = \tan^{-1}(4x^2)$. Find f'

$f'(x) = \frac{1}{1+(4x^2)^2} \cdot 8x$

$= \frac{8x}{1+16x^4}$

3.6 Derivatives of Logarithmic functions.

$\frac{d}{dx}(b^x) = \ln b \, b^x$, $b > 0, b \neq 1$

DIY $\frac{d}{dx}(\log_b x) = \frac{1}{x \ln b}$ (Use $y = \log_b x \Updownarrow x = b^y$)

$\frac{d}{dx}(\ln x) = \frac{d}{dx}(\log_e x) = \frac{1}{x \ln e} = \frac{1}{x}$

Now $\frac{d}{dx}[\ln(g(x))] = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)}$

ALSO $\frac{d}{dx}[\ln|x|] = \frac{1}{x} \rightarrow$ See Ex 6 Pg 220.

Ex $f(x) = \ln\left[\frac{(x+1)^2}{\sqrt{x-3}}\right]$, find $f'(x)$

$= \ln[x+1]^2 - \ln\sqrt{x-3}$

$= 2\ln(x+1) - \frac{1}{2}\ln(x-3)$

$f'(x) = 2 \cdot \frac{1}{x+1} - \frac{1}{2} \cdot \frac{1}{x-3}$

$= \frac{2}{x+1} - \frac{1}{2(x-3)}$

Logarithmic Diff

Ex $y = \dfrac{x^{3/4}\sqrt{x^2+1}}{(x^5+1)^5}$

1) Take ln of both sides and use property of logarithms to simplify.

$$\ln y = \ln\left[\frac{x^{3/4}\sqrt{x^2+1}}{(x^5+1)^5}\right]$$

$$\Rightarrow \ln y = \frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(x^5+1)$$

2) Do Implicit diff.

$$\frac{d}{dx}[\ln y] = \frac{d}{dx}\left[\frac{3}{4}\ln(x) + \frac{1}{2}\ln(x^2+1) - 5\ln(x^5+1)\right]$$

$$\Rightarrow \frac{y'}{y} = \frac{3}{4}\cdot\frac{1}{x} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}x}{(x^2+1)} - 5\cdot\frac{5x^4}{x^5+1}$$

$$\Rightarrow \frac{y'}{y} = \frac{3}{4x} + \frac{x}{(x^2+1)} - \frac{25x^4}{x^5+1}$$

3

$$\Rightarrow y' = \left[\frac{3}{4x} + \frac{x}{x^2+1} - \frac{25x^4}{x^5+1}\right]\cdot\frac{x^{3/4}\sqrt{x^2+1}}{(x^5+1)^5}$$

Ex $y = x^{\sqrt{x}}$, find y'

1) $\ln y = \ln(x^{\sqrt{x}})$

$\Rightarrow \ln y = \sqrt{x}\ \ln x$

2) $\dfrac{d}{dx}[\ln y] = \dfrac{d}{dx}[\sqrt{x}\ \ln x]$

$$\Rightarrow \frac{y'}{y} = \sqrt{x}\cdot\frac{1}{x} + \frac{1}{2\sqrt{x}}\ \ln x$$

$$\Rightarrow \frac{y'}{y} = \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}} = \frac{2+\ln x}{2\sqrt{x}}$$

$$\Rightarrow \frac{y'}{y} = \frac{2+\ln x}{2\sqrt{x}}$$

$$\Rightarrow \quad \frac{y'}{y} = \frac{2+\ln x}{2\sqrt{x}}$$

3) $$y' = \left[\frac{2+\ln x}{2\sqrt{x}}\right] \cdot x^{\sqrt{x}}$$

$\csc^{-1} x$

$\sec^{-1} x$

$\cot^{-1} x$